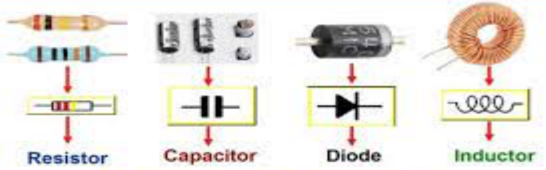
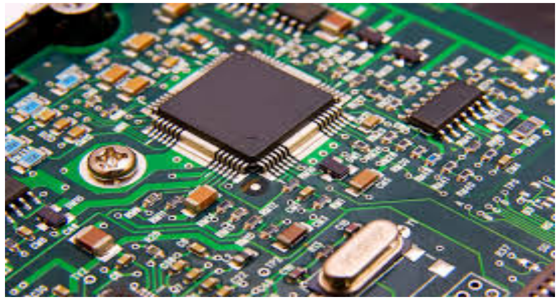
Electronics 1

BSC 113

Fall 2022-2023

Lecture 9





Natural and Step Response for RL, RC and RLC Circuit

INSTRUCTOR

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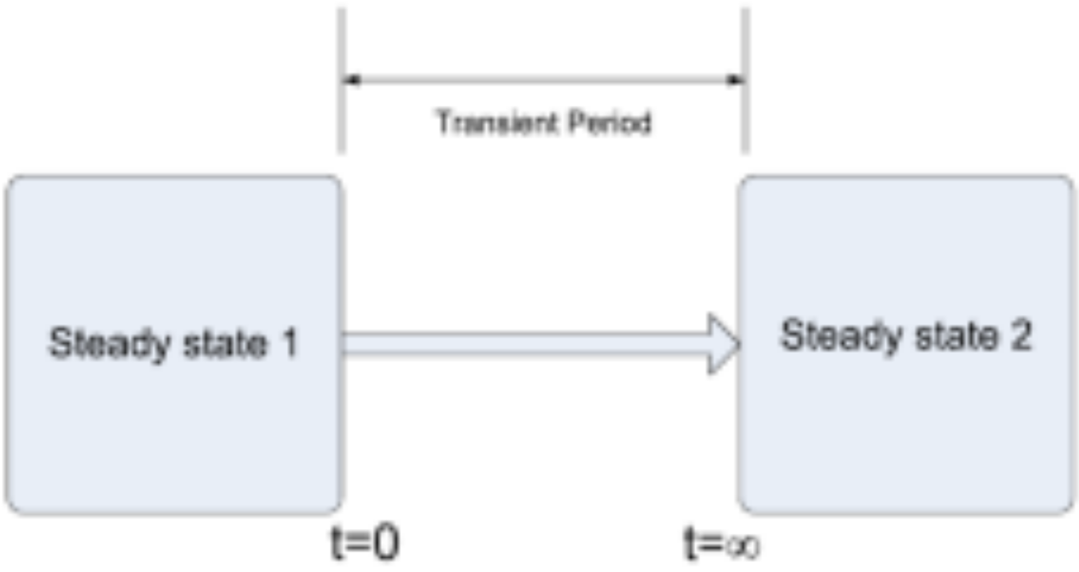
Contents

First order transient circuit

- In this chapter, we shall examine two types of simple circuits: a circuit comprising a resistor and capacitor and a circuit comprising a resistor and an inductor.
- These are called RC and RL circuits, respectively.
- A first-order circuit is characterized by a first-order differential equation.



1- Source free R-C circuit

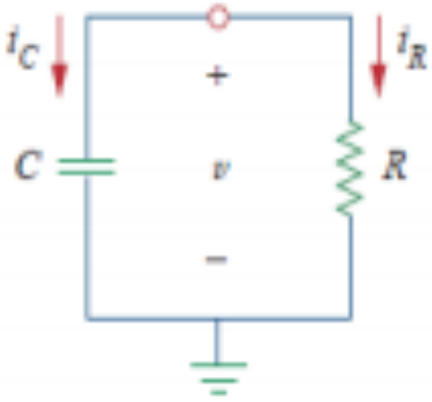
- A source-free RC circuit occurs when its dc source is suddenly disconnected.

- The energy already stored in the capacitor is released to the resistors (sometimes called free response).

- This shows that the voltage response of the RC circuit is an exponential decay of the initial voltage.

- Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the natural response of the circuit.

1- Source free R-C circuit



$$v(0) = V_0$$

$$w(0) = \frac{1}{2}CV_0^2$$

$$i_C + i_R = 0$$

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

$$\frac{dv}{v} = -\frac{1}{RC}dt$$

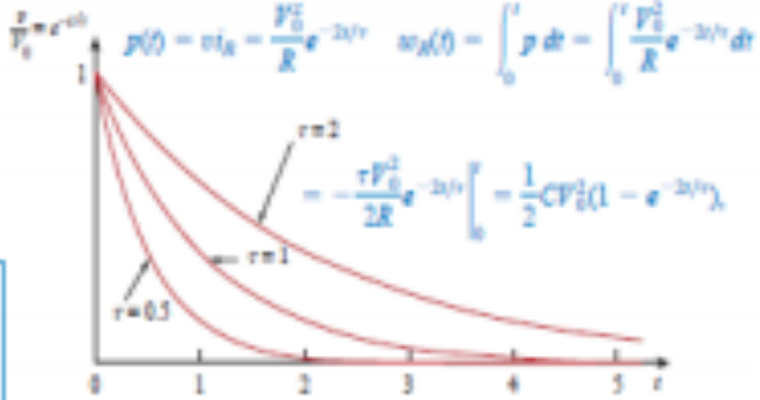
$$\ln\frac{v}{A} = -\frac{t}{RC}$$

$$v(t) = Ae^{-t/RC}$$

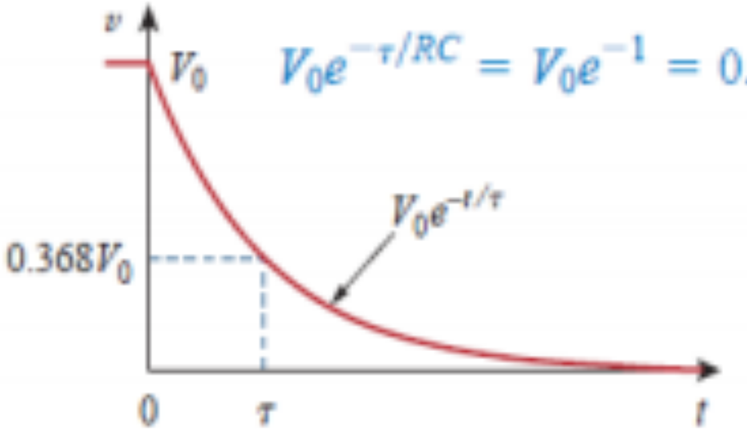
$$v(t) = V_0e^{-t/RC}$$

$$p(t) = vi_R = \frac{V_0^2}{R}e^{-2t/\tau} \qquad w_R(t) = \int_0^t p\,dt = \int_0^t \frac{V_0^2}{R}e^{-2t/\tau}dt$$

$$= -\frac{\tau V_0^2}{2R}e^{-2t/\tau}\Big|_0^t = \frac{1}{2}CV_0^2(1 - e^{-2t/\tau}),$$



$$V_0e^{-\tau/RC} = V_0e^{-1} = 0.368V_0$$

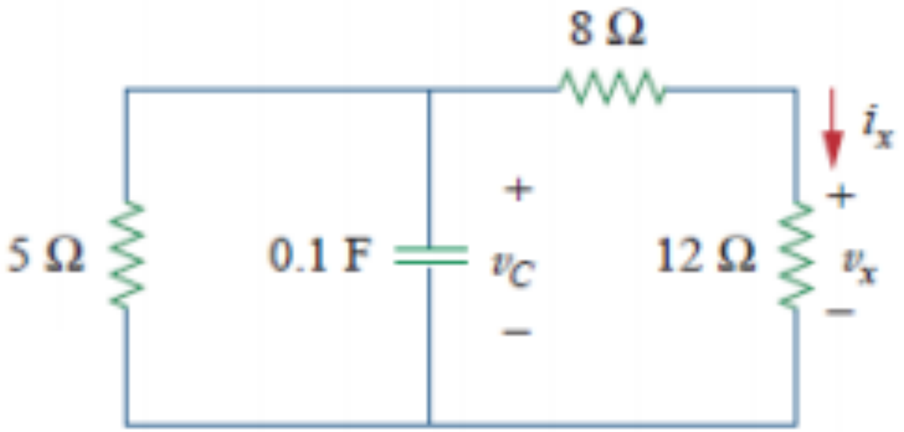


$$\tau = RC$$

$$v(t) = V_0e^{-t/\tau}$$

Example

- let $V_c(0)=15$, find V_c, V_x and i_x for $t>0$



$$R_{eq} = \frac{20 \times 5}{20 + 5} = 4\ \Omega$$

$$\tau = R_{eq}C = 4(0.1) = 0.4\ \text{s}$$

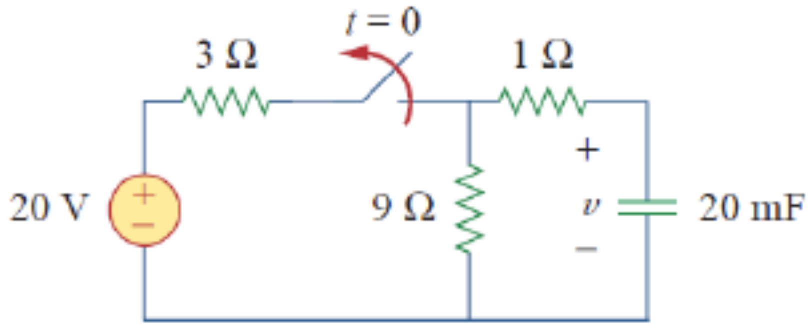
$$v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V}, \qquad v_C = v = 15e^{-2.5t}\ \text{V}$$

$$v_x = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}\ \text{V}$$

$$i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$$

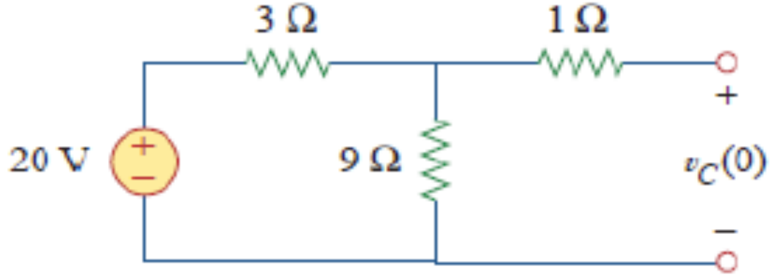
Example

- the switch in the circuit in the following Fig. has been closed for a long time, and it is opened at t =0. Find v(t) for t ≥ 0. Calculate the initial energy stored in the capacitor.



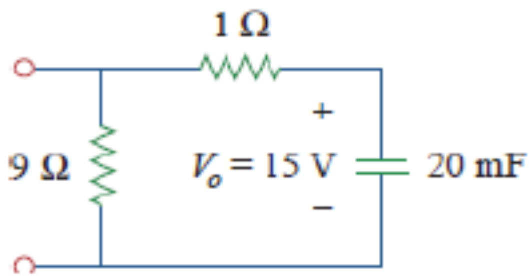
Answer: For t < 0 the switch is closed; the capacitor is an open circuit to dc, as represented in the following figure. Using voltage division

Vc(o) = 20*9/12 =15V



Example

- For the switch is opened, and we have the RC circuit shown in the following figure.



[Notice that the RC circuit is source free] The resistors in series give

$$R_{eq} = 9 + 1 = 10$$

The time constant is

$$\tau = R_{eq} * C = 0.2s$$

Thus, the voltage across the capacitor for is

$$v(t) = 15e^{-5t}V$$

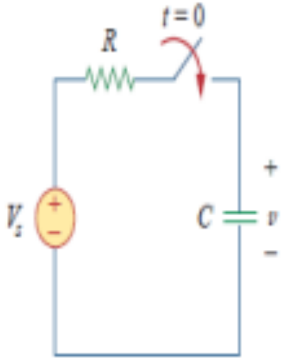
The initial energy stored in the capacitor is

Wc(0)=0.5 C*V²=2.25 J

2- Forced R-C circuit

- ➢ The steady-state response is the behavior of the circuit a long time after an external excitation is applied.

- ➢ The forced R-C circuit is shown in figure. with v(∞) is the final value of capacitor voltage.



$$v(0^-) = v(0^+) = V_0$$

$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0$$

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

$$\frac{dv}{dt} = -\frac{v - V_s}{RC}$$

$$\frac{dv}{v - V_s} = -\frac{dt}{RC}$$

$$\ln(v - V_s)\Big|_{V_0}^{v(t)} = -\frac{t}{RC}\Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

$$\ln\frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

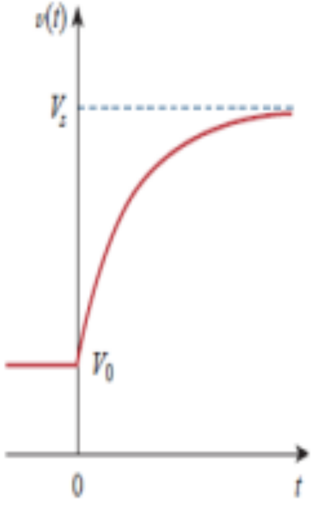
$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \quad \tau = RC$$

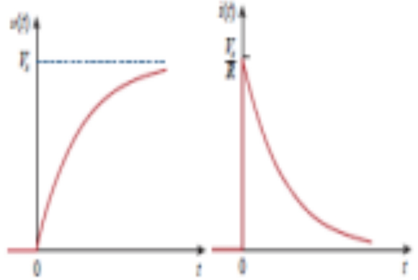
$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0$$

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

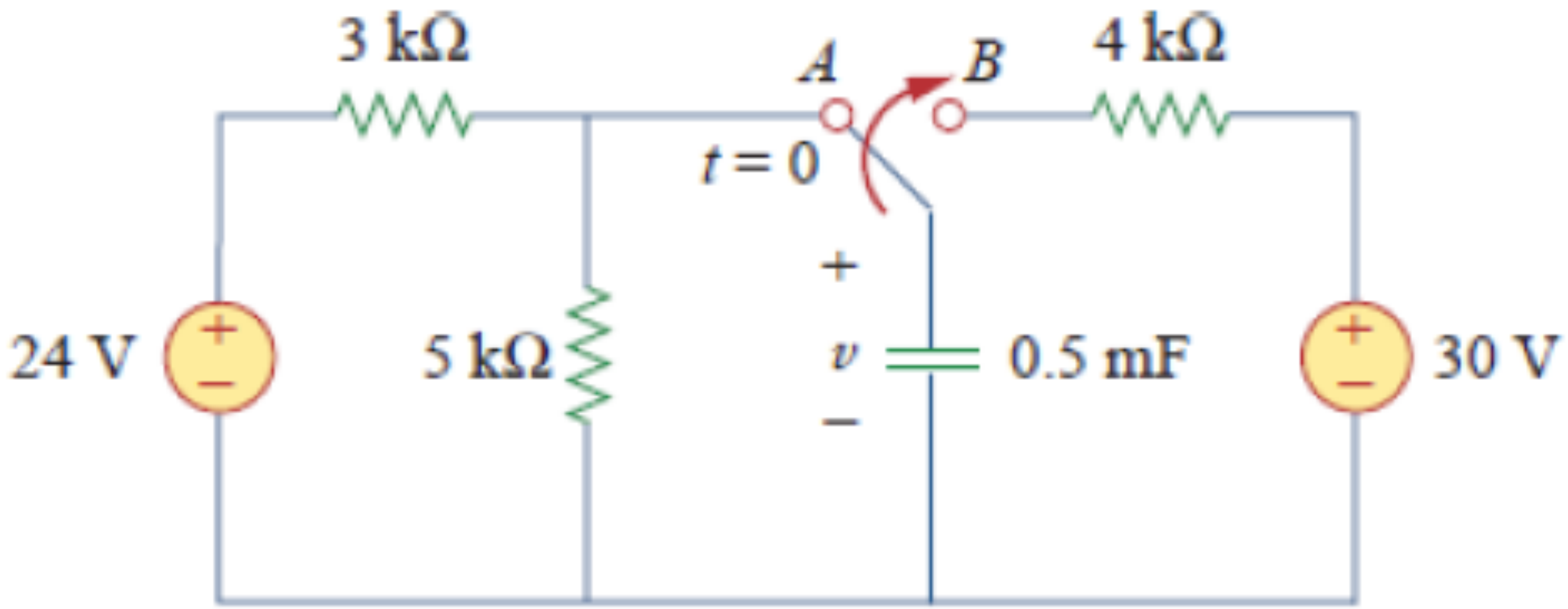
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$





Example

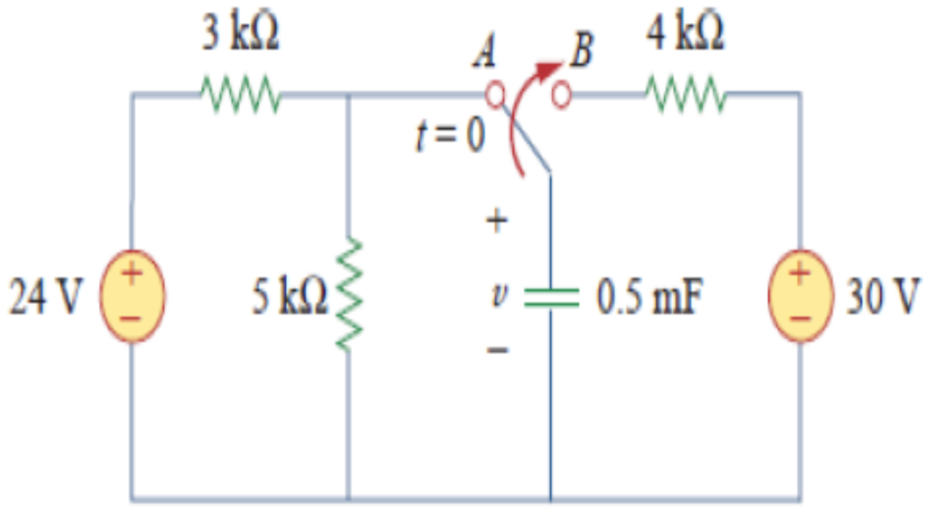
- the switch in the following Fig. has been in position A for a long time, and it is moved to B at t =0. Determine v(t) for t > 0. Calculate its value at t =1s and t = 4s.



Example

- For t < 0 the switch is at position A. The capacitor acts like an open circuit to dc, but v is the same as the voltage across the resistor 5k. Hence, the voltage across the capacitor just before t=0 is obtained by voltage division as using the fact that the capacitor voltage cannot change instantaneously,

$$v(0) = \frac{5}{5+3}24 = 15$$



For t>0, the switch is in position B. The Thevenin resistance connected to the capacitor is $R_{Th} = 4\ k$ and the time constant is

$$\tau = R_{eq} * C = 2s$$

Since the capacitor acts like an open circuit to dc at steady state, $v(\infty) = 30$ V

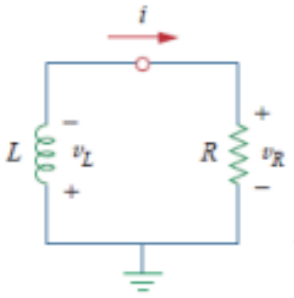
$$v(t) = 30 - 15e^{-0.5t}V$$

at t=1 v=20.9 V

at t=4 v=27.97V

3- Source free R-L circuit

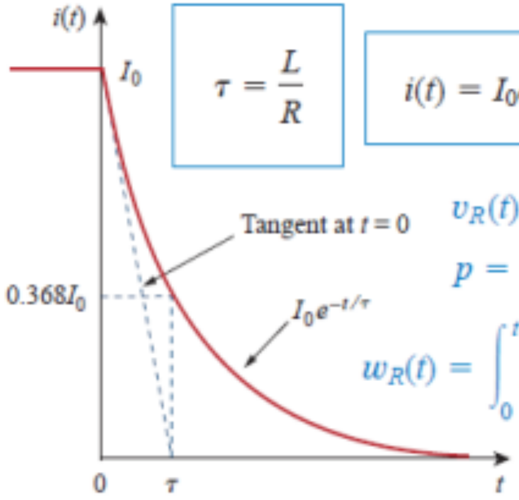
- Consider the series connection of a resistor and an inductor, as shown in Fig.
- Our goal is to determine the circuit response.



$$i(0) = I_0$$
$$w(0) = \frac{1}{2} L I_0^2$$
$$v_L + v_R = 0$$
$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0$$
$$\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\ln i \Big|_{I_0}^{i(t)} = -\frac{Rt}{L}\Big|_0^t$$
$$\ln \frac{i(t)}{I_0} = -\frac{Rt}{L}$$
$$i(t) = I_0 e^{-Rt/L}$$



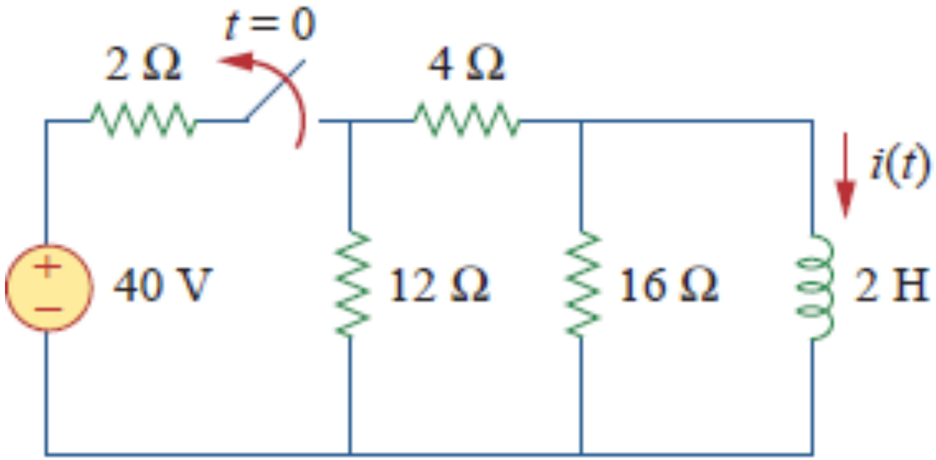
$$\tau = \frac{L}{R}$$

$$i(t) = I_0 e^{-t/\tau}$$

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$
$$p = v_R i = I_0^2 R e^{-2t/\tau}$$
$$w_R(t) = \int_0^t p\, dt = \int_0^t I_0^2 R e^{-2t/\tau}\, dt = -\frac{1}{2}\tau I_0^2 R e^{-2t/\tau}\Big|_0^t$$
$$w_R(t) = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau})$$

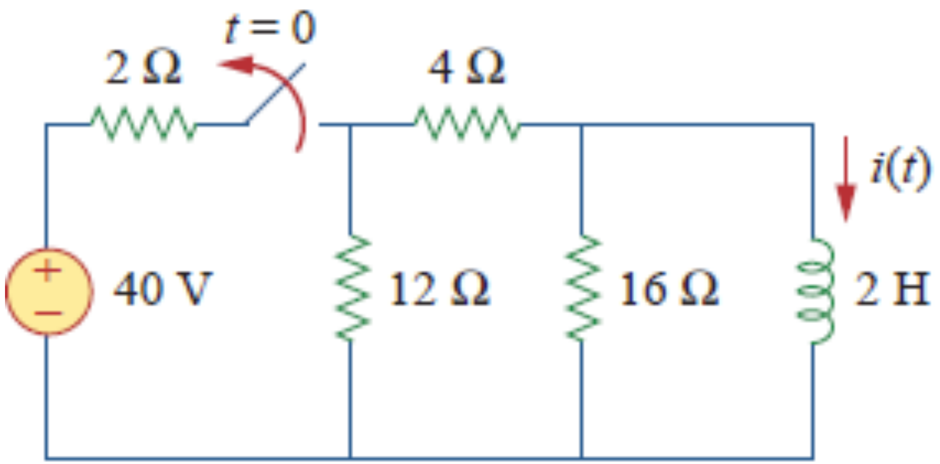
Example

- the switch in the circuit in the following Fig. has been closed for a long time, and it is opened at t =0. Find i(t) for t ≥0



When t < 0, the switch is closed, and the inductor acts as a short circuit to dc. The resistor 16 is short-circuited; the resulting circuit is shown. To get $i1$in circuit, we combine the and resistors in parallel.

Example



$$i_1 = \frac{40}{2 + \frac{4 * 12}{12 + 4}} = 8A$$

We obtain i (0) from i_1 using current division, by writing

$$i(t) = \frac{12}{12 + 4} i_1 = 6A$$

When t > 0, the switch is open and the voltage source is disconnected. We now have the source-free RL circuit. Combining the resistors, we have

$$R_{eq} = (12 + 4)//16 = 8\Omega$$

The time constant is

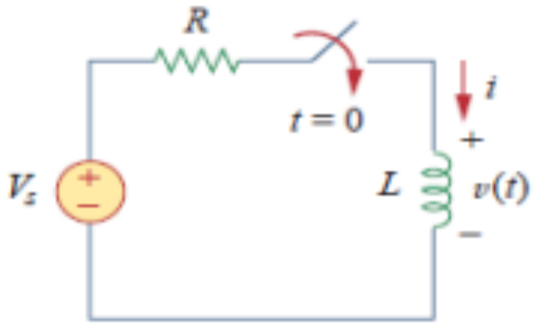
$$\tau = \frac{L}{R_{eq}} = 0.25\ s$$

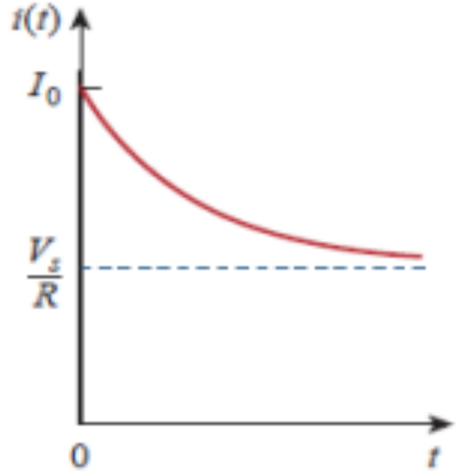
Thus,

$$i(t) = 6e^{-4t} A$$

4 Forced R-L circuit

- The steady-state response is the behavior of the circuit a long time after an external excitation is applied. The forced R-L circuit is shown in figure with i (∞) is the final value of inductor current.





$$i = i_t + i_{ss}$$

$$i_t = Ae^{-t/\tau}, \qquad \tau = \frac{L}{R}$$

$$i_{ss} = \frac{V_s}{R}$$

$$i = Ae^{-t/\tau} + \frac{V_s}{R}$$

$$i(0^+) = i(0^-) = I_0$$

$$I_0 = A + \frac{V_s}{R}$$

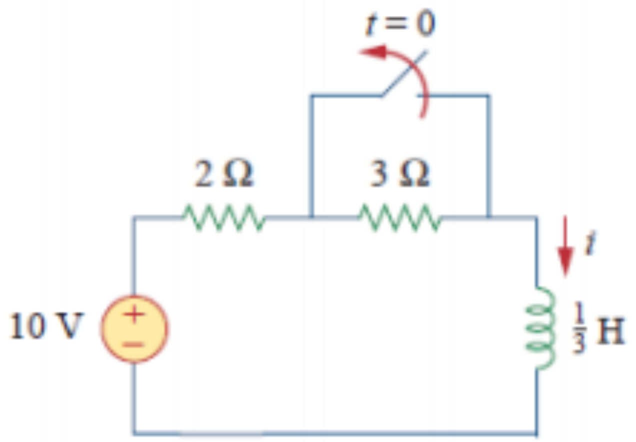
$$A = I_0 - \frac{V_s}{R}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau}$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

Example

- Find i(t) in the circuit in the following Fig. for t >0. Assume that the switch has been closed for a long time.



Answer: When t < 0 the 3-ohm resistor is short-circuited, and the inductor acts like a short circuit. The current through the inductor is

$$i(0) = \frac{10}{2} = 5A$$

When t > 0 the switch is open.

$$i(\infty) = \frac{10}{2+3} = 2A$$

The Thevenin resistance across the inductor terminals is

$$R_{Th} = 2 + 3 = 5$$

Thus,

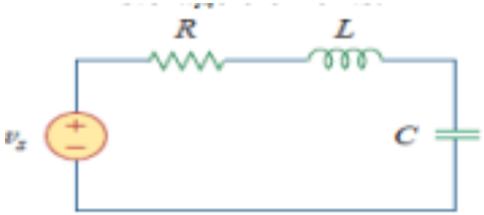
$$i(t) = 2 + 3e^{-15t}A$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

Second order transient circuit

- In this chapter we will consider circuits containing two storage elements. These are known as second-order circuits because their responses are described by differential equations that contain second derivatives.
- Typical examples of second-order circuits are RLC circuits, in which the three kinds of passive elements are present.
- Examples of such circuits are shown in figure. A second-order circuit is characterized by a second-order differential equation.
- It consists of resistors and the equivalent of two energy storage elements.

Second order transient circuit



$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$$

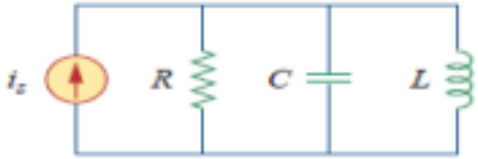
$$i(0) = I_0$$

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$



$$i = Ae^{st}$$

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

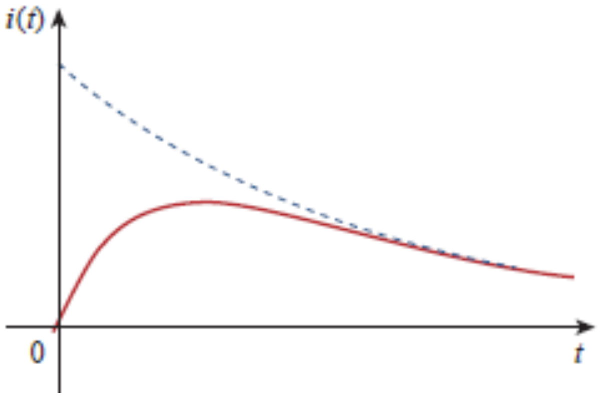
$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Second order transient circuit

- There are three types of solutions as shown in figure :
  - If α > $w0$ we have the over-damped case.
  - If α = $w0$we have the critically damped case.
  - If α < $w0$ we have the under-damped case.

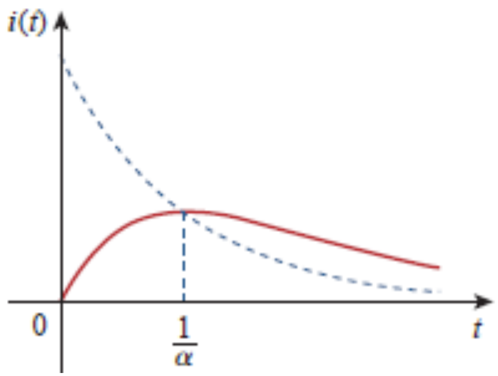
Overdamped Case ($\alpha > \omega_0$)

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



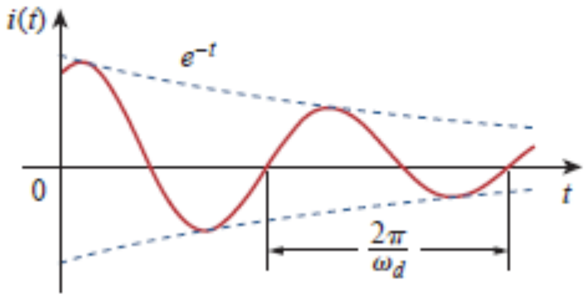
Critically Damped Case ($\alpha = \omega_0$)

$$i(t) = (A_2 + A_1 t)e^{-\alpha t}$$



Underdamped Case ($\alpha < \omega_0$)

$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$



Second order transient circuit

- If we have step response

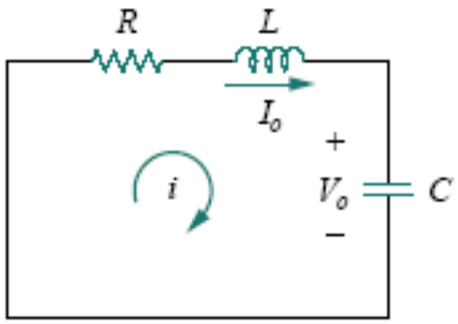
$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}$$

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)}$$

Example

- In the following figure R= 40-ohm, L = 4 H and C= 0.25 F. Calculate the characteristics roots of the circuit. Is the natural response over damped, under damped and critical damped?



Answer: $\alpha = R/2L = 5$ and $w_0 = \frac{1}{\sqrt{LC}} = 1$

$\alpha > w_0$ the circuit is over-damped $S_1 = -0.101$ and $S_2 = -9.899$


Thank you